THE NATURE OF POTENTIALS AND GAUGE TRANSFORMATIONS IN CLASSICAL ELECTROMAGNETICS

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Scope

- This work deals with a meaning of the potentials and gauge transformations in classical electromagnetics.
- In majority of EM course textbooks Maxwell equations are postulated from the empirical basis featuring the use of electric and magnetic fields as quantities of interest.
- EM potentials are treated as ***auxilliary mathematical functions*** being neither unique nor measurable, thus not having any physical meaning.

Scope

- The problem of uniqueness is handled by gauge transformations.
- The choice of different gauge conditions is often considered to be governed by pure mathematical conveniences not affecting the electric and magnetic fields.
- These fields are then regarded as gauge invariant.

Scope

- In modern physics the ***principle of gauge invariance*** is considered to be the keystone for any physical field.
- From this view the gauge conditions could be regarded as ***continuity equations in electromagnetics.***
- This obvious ambiguity and dichotomy have become a rather hot topic in both history and philosophy of physics.

The vector potential is the mathematical quantity which can be considered as the fundamental quantity of the electromagnetic theory.

James Clerk Maxwell

Introduction

- In classical electromagnetics the potentials are not regarded as unique quantities and, therefore, are not measurable physical quantities.
- They are viewed as merely mathematical constructs not represent physically existing fields.
- The electric field and magnetic field can be readily defined in terms of potentials *A* and φ as they are invariant under certain gauge transformations.
- Therefore, contrary to the potentials, the fields are uniformly determined.

Introduction

- Invariance, or symmetry, represents a change in system which does not affect the action integral, or the equation of motions while gauge principle is considered to be a central concept in fundamentals of theoretical physics.
- In classical electromagnetics one starts from Maxwell's equations treating them as mathematical representations of experimentally discovered natural laws.
- The ***continuity equation*** is then considered as a consequence of ***Maxwell's equations***.

Introduction

- Within this approach ***the fields are gauge invariant***, while ***potentials are just auxiliary functions*** - pure mathematical constructs without proper physical meaning.

- An opposite approach is mathematically also possible, i.e. if one exploits symmetry properties of the Lagrangian, and thus easily introduces potentials.

- Using such an approach it is possible to derive continuity equation, Lorenz force and Maxwell'e equations from a proper Lagrangian.

Introduction

- In quantum physics ***A*** represents a fundamental quantity in the Schrodinger equation for a charged particle and in interactions occurring in quantum electrodynamics.
- Some authors addressed certain experiments demonstrating the reality and importance of potentials in quantum physics.
- Within the famework of theory of relativity featuring covariant formulation magnetic vector potential is composed with scalar potential into the four potential.

Introduction

- The problem of physical meaning of ***scalar potential*** is appreciably less pronounced as it can be easily understood as ***potential energy per unit charge.***
- J. C. Maxwell considered vector potential to be a ***stored momentum per unit charge*** and named it ***electromagnetic momentum*****.**
- Thomson shared a similar attitude and considered vector potential as appropriate ***field momentum per unit charge.***

Introduction

- Nowadays dominant view in classical electromagnetics textbooks came from Heaviside and Hertz.
- Independently from each other they rewrote the original 20 scalar Maxwell's equations into modern vector form.
- They both treated vector potential as nonphysical, artificial quantities convenient only for easier calculations of physicaly existing electric and magnetic fields.

Introduction

- This work discusses a possible meaning of *magnetic vector potential* and *Lorenz gauge* by which vector potential is mathematically completely determined.

- Equation of continuity, expressing the conservation of charge, stemming from symmetry of the Lagrangian in classical mechanics is addressed and then Lorenz gauge is discussed.

- It is shown that ***Lorenz gauge*** can be considered as a ***continuity equation for potentials*** and how this gauge is associated with equation of continuity for charge and current density, respectively.

Continuity equation and vector potential from gauge invariance

- Lagrangian *L* in classical mechanics is defined as difference between kinetic energy (W_{kin}) and potential energy (W_{pot}) of the system.

$$L = \frac{1}{2} m \dot{r}^2 - W_{pot}(\vec{r})$$

- According to the symmetry of the Lagrangian a total time derivative may be added to *L* without changing the equation of motion, one can define a new Lagrangian *L'*

$$L' = L + q \frac{d\Lambda}{dt}$$

Continuity equation and vector potential from gauge invariance

- For the point particle the charge density ρ can be written

$$\rho\left(\vec{R},t\right)=q\delta\left[\vec{R}-\vec{r}(t)\right]$$

- In the next step, the current density can be expressed as charge in motion, i.e.

$$\vec{J}\left(\vec{R},t\right)=q\frac{d\vec{r}(t)}{dt}\delta\left[\vec{R}-\vec{r}(t)\right]$$

- Integrating over volume *V* yields

$$q=\int_V \rho\left(\vec{R},t\right)dV$$

Continuity equation and vector potential from gauge invariance

- Performing total differentiation with respect to time

$$L' = L + q\left[\frac{\partial\Lambda}{\partial n}\frac{dr}{dt} + \frac{\partial\Lambda}{\partial t}\right]$$

Lagrangian (2) can be written as follows

$$L_0^{'} = L_0 + \partial_\mu\left(J^\mu\Lambda\right)$$

where: $\partial_\mu = \dfrac{\partial}{\partial x^\mu} \quad \mu = 0,1,2,3 \quad \partial_\mu \Rightarrow \left(\dfrac{\partial}{\partial x^0}, \dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^2}, \dfrac{\partial}{\partial x^3}\right)$

- Note that 0 –komponent pertains to time, while 1, 2 and 3 denote *x*, *y* and *z* component.

Continuity equation and vector potential from gauge invariance

J^{μ} - current four-vector assigned to the particle
Λ – arbitrary function of x^{μ}

- It can be written:

$$J^{\mu} = \left(\rho, J^{x}, J^{y}, J^{z}\right)$$

- As the zero label pertains to time component, it follows

$$J^{0} = \rho$$

where ρ is the volume charge density.

Continuity equation and vector potential from gauge invariance

- Furthermore, it can be written

$$L_0^{'} = L_0 + \Lambda \partial_\mu J^\mu + J^\mu \partial_\mu \Lambda$$

- Now a new four-vector (four-potential) can be introduced

$$\left(A_1, A_2, A_3, A_4\right) = \left(\varphi, -A\right)$$

and the new Lagrangian can be written as follows

$$L_{0i} = L_i - J^\mu A_\mu$$

Continuity equation and vector potential from gauge invariance

- One has

$$L_{0i}^{'} = L_0 + \Lambda \partial_\mu J^\mu - J^\mu A_\mu^{'}$$

where

$$A_\mu^{'} = A_\mu - \partial_\mu \Lambda$$

which can be wriiten as following set of equations:

$$\vec{A}' = \vec{A} + \nabla\Lambda\left(\vec{R},t\right) \qquad \varphi' = \varphi - \frac{\partial\Lambda\left(\vec{R},t\right)}{\partial t}$$

where *A* and *A'* stand for magnetic vector potential, while φ denotes the electric scalar potential.

Continuity equation and vector potential from gauge invariance

- Therefore, *A'* and *A* satisfy all the equations and result in same (physically existing) fields.
- A_μ is not determined by any prescribed initial condition, therefore, a part of A_μ, i.e. its one degree of freedom does not have a physical meaning.
- This ***spurious degree of freedom*** can be eliminated by imposing a constraint, or so-called gauge condition, such as the Lorenz gauge.

Continuity equation and vector potential from gauge invariance

- Now, Lagrangians L_0 and L'_{0i} are equivalent if the following condition is satisfied

$$\partial_\mu J^\mu = 0$$

which can be written in the standard form

$$\nabla \cdot \vec{J}\left(\vec{R},t\right) - \frac{\partial \rho\left(\vec{R},t\right)}{\partial t} = 0$$

- This is *equation of continuity* relating charge and current densities, derived from the gauge invariance of classical mechanics (symmetry property of the Lagrangian).

Continuity equation and vector potential from gauge invariance

- Conservation of electric charge is a consequence of the Lagrangian gauge invariance.

- The same result could be obtained from total electromagnetic Lagrangian

$$L_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J_i A_i - \rho\varphi$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Continuity equation and vector potential from gauge invariance

- Faraday tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- In the standard vector notation the total electromagnetic Lagrangian density can be written as

$$L_d = \frac{1}{2\mu}\cdot\left(\nabla\times\vec{A}\right)^2 - \frac{1}{2}\cdot\varepsilon\left(\nabla\varphi + \frac{\partial\vec{A}}{\partial t}\right)^2 + \vec{J}\cdot\vec{A} - \varphi\cdot\rho$$

- Lagrangian density can be easily shown to contain four Maxwell equations.

The *meaning* of vector potential

- Magnetic vector potential is to a certain extent associated with total momentum of the charged particle.

- For a charged particle moving along one axis (i) of rectangular coordinate system by a velocity v_i a Lagrangian can be written in the form

$$L = \frac{1}{2} m v_i^2 - W_{pot} + q v A_i - q\varphi$$

where W_{pot} and $q\varphi$ are different contributions to potential energy.

The *meaning* of vector potential

- The canonical momentum p_i is defined as

$$p_i = \frac{\partial L}{\partial v_i}$$

where $p_i = mv_i + qA_i$ can be regarded as generalized momentum being conserved under certain conditions.

- This generaliezd momentum is now composed from well-established ***linear momentum mv*$_i$** and quantity qA_i which can be referred to as EM momentum which is, in accordance to the *Maxwell-Thomson* view of magnetic vector potential, ***the stored momentum per unit charge.***

The *meaning* of Lorentz gauge

- Expressing the electric and magnetic fields in terms of scalar and vector potential

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \varphi$$

Gauss law for the electric field

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

can be now written

$$\nabla^2 \varphi + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\frac{\rho}{\varepsilon}$$

The *meaning* of Lorentz gauge

- In addition, from the generalized Ampere's law

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

one obtains

$$-\nabla \times \nabla \times \vec{A} = -\mu \vec{J} + \mu\varepsilon \left[\nabla \left(\frac{\partial \varphi}{\partial t} \right) + \frac{\partial^2 \vec{A}}{\partial t^2} \right]$$

- Taking into account identity $\nabla \times \nabla \times \vec{A} = \nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}$

It follows

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla \left(\nabla \cdot \vec{A} \right) + \mu\varepsilon \nabla \left(\frac{\partial \varphi}{\partial t} \right)$$

The *meaning* of Lorentz gauge

- Now, it follows

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla\left(\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} \right)$$

- Choosing Lorentz gauge

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0$$

one deals with wave equation for potentials.

- In covariant formalism one has

$\partial^{\mu} A_{\mu} = 0$ or $\partial_{\mu} A^{\mu} = 0$ where $\partial^{\mu} = \left(\frac{\partial}{\partial t}, \nabla \right)$ $\partial_{\mu} = \left(\frac{\partial}{\partial t}, -\nabla \right)$

The *meaning* of Lorentz gauge

- Therefore, Lorentz gauge leads to the set of nonhomogeneous wave equations for scalar and vector potential, respectively:

$$\nabla^2\varphi - \mu\varepsilon\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} \qquad \nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

- Now, assuming that *J* and *ρ* represent all sources within a volum *V*' the solution of of the potential wave equations can be expressed by following particular integrals:

$$\vec{A}(r,t) = \frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}\,',t-R/c)}{R}dV' \quad \varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(\vec{r}\,',t-R/c)}{R}dV'$$

The *meaning* of Lorentz gauge

- The variables pertaining to source and observation point, respectively, are shown in Fig 1.

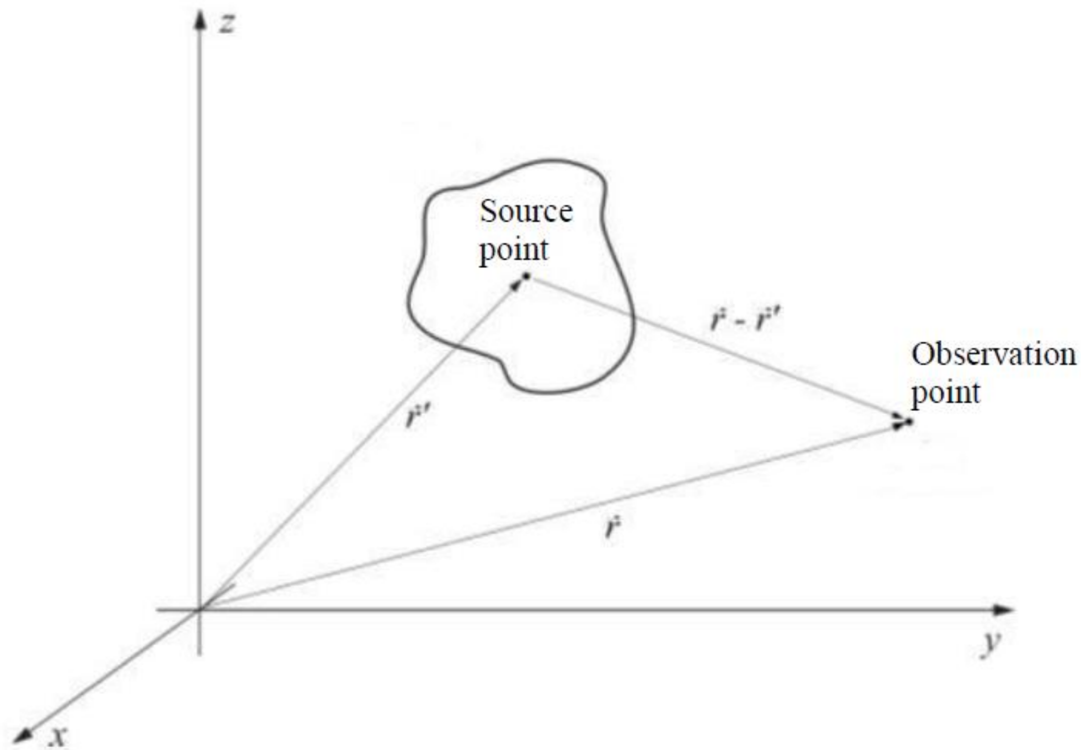

Fig.1. The source point and observation point

The *meaning* of Lorentz gauge

- What is the meaning of Lorenz gauge?

$$\nabla\cdot\vec{A}+\mu\varepsilon\frac{\partial\varphi}{\partial t}=0$$

- To come up with an answer it is helpful to write a counterpart in in the frequency domain

$$\nabla\cdot\vec{A}+j\omega\mu\varepsilon\cdot\varphi=0$$

- For time-harmonic dependence set of equations the retarded potentials simplfy into

$$\varphi(\vec{r})=\frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(\vec{r}\,')e^{-jkR}}{R}dV' \qquad \vec{A}(r)=\frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}\,')e^{-jkR}}{R}dV'$$

The *meaning* of Lorentz gauge

- Now, one has

$$\frac{\mu}{4\pi}\int_{V'} \nabla\cdot\vec{J}(\vec{r}\,')\frac{e^{-jkR}}{R}dV' + \frac{j\omega\mu}{4\pi}\int_{V'}\rho(\vec{r}\,')\frac{e^{-jkR}}{R}dV' = 0$$

which can be written

$$\int_{V'}\left[\nabla\cdot\vec{J}(\vec{r}\,') + j\omega\rho(\vec{r}\,')\right]\frac{e^{-jkR}}{R}dV' = 0$$

or, using the Green finction notation one has

$$\int_{V'}\left[\nabla\cdot\vec{J}(\vec{r}\,') + j\omega\rho(\vec{r}\,')\right]G\left(\vec{r},\vec{r}\,'\right)dV' = 0$$

The *meaning* of Lorentz gauge

- Now, as $G(\vec{r},\vec{r}\,')$ is always different from zero one obtains

$$\nabla \cdot \vec{J}(\vec{r}\,') + j\omega\rho(\vec{r}\,') = 0$$

which is a frequency domain counterpart of

$$\nabla \cdot \vec{J}\left(\vec{R},t\right) - \frac{\partial \rho\left(\vec{R},t\right)}{\partial t} = 0$$

The *meaning* of Lorentz gauge

- Therefore, equation

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0$$

or its frequency domain counterpart

$$\nabla \cdot \vec{A} + j\omega\mu\varepsilon \cdot \varphi = 0$$

essentially represents the continuity equation for potentials.

- This could be regarded as a 'missing' physical meaning of Lorentz gauge one may look for.

Conclusion

- The nature of the potentials and gauge transformations in classical electromagnetics is discussed in this work.
- In the most commonly used approach in textbooks on electromagnetics Maxwell equations dealing with the fields and their sources are postulated from the empirical basis.
- Within such an approach the potentials are derived from Maxwell's equations as purely mathematical construct aiming to provide one with more efficient calculation tool for fields.

Conclusion

- These potentials are neither unique nor measurable quantities and, consequently, are not considered to have a physical meaning.
- The problem of uniqueness is handled by gauge transformations, while different gauge conditions are often considered as pure mathematical conveniences not affecting the electric and magnetic fields.

Conclusion

- In addition to the fact that the principle of gauge invariance is considered to be crucial for the rigorous mathematical description of fields in contemporary physics it is also shown that ***Lorenz gauge*** could be regarded as ***continuity equation for potentials.***

- Finally, physical meaning of ***magnetic vector potential*** is associated with ***electromagnetic momentum***, while ***scalar potential*** is regarded as ***energy per unit charge.***

Thank you for your kind attention!

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