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ON THE STORED ENERGY IN THE ANTENNA ELECTROMAGNETIC FIELD AND CORRESPONDING LIMITS IN THE TRANSFER OF INFORMATION

Abstract: The paper reviews some fundamental concepts of the energy conservation in the electromagnetic field stemming from Hamilton principle in electromagnetics and Poynting theorem with particular emphasis to the analysis of the correlation of the stored energy in the antenna electromagnetic field with fundamental limits arising from the antenna dimensions. Of particular interest is to address correlation between electric/physical dimensions of the antenna and Q-factor and radiation efficiency, which is crucial for the assessment of maximum information transfer. Presented theoretical concepts are illustrated for the case of Hertz dipole.

Keywords: Hamilton principle, Poynting theorem, stored energy, fundamental limits of antennas

Introduction

Fundamental limits in antenna operation deal with antenna dimensions and relationship between its stored and irradiated energy [1-2], respectively. Though there is a sort of correlation of stored energy and dissipated (radiated) energy in antenna theory with energy usable for useful work and reactive energy, or what is more in use, active power and reactive power, there are some important distinctions and related physical consequences, as has been discussed in [2]. On the other hand, the most rigorous approach to address the issue of antenna behavior is to use Hamilton principle and the Poynting theorem expressing the conservation law of the electromagnetic field energy. An important engineering parameter in assessing the fundamental limits of antennas is the quality factor (Q-factor) connected with actual antenna bandwidth. There are different expressions for Q-factor in literature but, for time-harmonic radiators, such as Hertz dipole, its definition generally pertains to the ratio of stored energy and dissipated energy per cycle [3]. There are also some controversies regarding the definition of stored energy in the vicinity of the antenna [1-2].

Various aspects of this issue have been of interest starting from the very beginning of radiocommunications from early 20th century to the contemporary issues pertaining to wireless communications and Internet of Things (IoT) [1].

This paper revisits the derivation of the integral form of Poynting theorem for time-harmonic dependent quantities. Next, for the case of radiation of Hertz dipole in free space the calculation of Q-factor is carried out using the apparent (complex) power defined as the surface integral over complex Poynting vector.

Thus, the Q-factor is derived simply as a ratio between imaginary and real part of complex power, respectively as suggested in [1]. It is worth noting that complex power directly arises from the integral form of Poynting theorem.

1. General Law of Conservation of Electromagnetic Energy – Poynting theorem

Using the principle of least action in classical electromagnetics, that is by minimization of corresponding action integral the of Maxwell equations can be obtained. Furthermore, the general conservation law of energy in the macroscopic electromagnetic field can be readily derived from curl Maxwell equations.

Starting from divergence of Poynting vector

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} \tag{1}$$

where E and H denotes the electric and magnetic field, respectively, while D and B represents electric and magnetic flux density, respectively.

Combining (1) with first two curl Maxwell equations yields:

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\frac{\partial}{\partial t}\left(\frac{\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}}{2}\right) - \vec{E} \cdot \vec{J} \tag{2}$$

Taking the volume integral over (2) it follows

$$\int_V \nabla \cdot (\vec{E} \times \vec{H})\, dV = -\frac{\partial}{\partial t}\int_V \left(\frac{\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}}{2}\right) dV - \int_V \vec{E} \cdot \vec{J}\, dV \tag{3}$$

For a battery with a non-electrostatic field *E'* the corresponding current density *J* can be written:

$$\vec{J} = \sigma(\vec{E} + \vec{E}') \quad (4)$$

Furthermore, applying the Gauss integral theorem to the left-hand side term, the volume integral transforms to the surface integral over the boundary, where $d\vec{A}$is the outward drawn normal vector surface element, that is one obtains

$$\int_V \vec{E}' \cdot \vec{J} dV = \frac{\partial}{\partial t} \int_V \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) dV + \int_V \frac{|\vec{J}|}{\sigma} \vec{J} dV' + \oint_A (\vec{E} \times \vec{H}) \cdot d\vec{A} \quad (5)$$

The sources within the volume of interest are balanced with the rate of increase of electromagnetic energy in the volume, the rate of flow of energy in through the domain surface and the Joule heat production in the domain.

For the time-harmonic quantities the complex Poynting vector is given by

$$\vec{S} = \frac{1}{2} (\vec{E} \times \vec{H} *) \quad (6)$$

and taking the divergence of Poynting vector yields

$$\nabla \cdot \vec{S} = \frac{1}{2} (\vec{H} * \nabla \times \vec{E} - \vec{E} \nabla \times \vec{H} *)$$

(7)

The divergence of complex power density can be expressed in terms of rate of stored energy, power lossess and sources, as follows:

$$\nabla \cdot \vec{S} = -j \frac{\omega}{2} \left(\mu |\vec{H}|^2 - \varepsilon |\vec{E}|^2 \right) - \frac{|\vec{J}|}{\sigma} + \frac{1}{2} \sigma |\vec{E}'|^2 \quad (8)$$

Integrating over a volume of interest one obtains

$$\int_V \nabla \cdot \vec{S} \, dV = -j \frac{\omega}{2} \int_V \left(\mu |\vec{H}|^2 - \varepsilon |\vec{E}|^2 \right) dV = -\frac{1}{2} \int_V \left| \frac{\vec{J}}{\sigma} \right|^2 dV + \frac{1}{2} \int_V \sigma |\vec{E}'|^2 \, dV \quad (9)$$

And applying the Gauss theorem it follows

$$\frac{1}{2} \int_A \vec{E} \times \vec{H} * d\vec{A} = -j \frac{\omega}{2} \int_V \left(\mu |\vec{H}|^2 - \varepsilon |\vec{E}|^2 \right) dV =$$
$$= -\frac{1}{2} \int_V \left| \frac{\vec{J}}{\sigma} \right|^2 dV + \frac{1}{2} \int_V \sigma |\vec{E}'|^2 \, dV \quad (10)$$

It is convenient to separate the real part and imaginary part of (10), that is the Poynting flow can be written as follows:

$$\frac{1}{2} Re \int_A \vec{E} \times \vec{H} * d\vec{A} = -\frac{1}{2}\int_V \left|\frac{\vec{J}}{\sigma}\right|^2 dV + \frac{1}{2}\int_V \sigma \left|\vec{E}'\right|^2 dV \qquad (11)$$

$$\frac{1}{2} Im \int_A \vec{E} \times \vec{H} * d\vec{A} = -\frac{\omega}{2}\int_V \left(\mu\left|\vec{H}\right|^2 - \varepsilon\left|\vec{E}\right|^2\right) dV \qquad (12)$$

The real part of the integral over Poynting vector represents the total average power while the imaginary part of the integral over Poynting vector is proportional to the difference between average stored magnetic energy in the volume and average stored energy in the electric field.
The ½ factor appears because *E* and *H* fields represent peak values, and it should be omitted for root-mean-square (*rms*) values.
The total average power can, for example represent the radiated power by an antenna. In addition, the first volume integral on the right-hand side of (11) represents power loss in the conduction currents and it just twice the average power loss.

2. Determination of the Q-factor

The complex (apparent) power P_s, according to the notation of circuit theory, is given by left-hand side of (10):

$$P_s = \frac{1}{2}\int_A \vec{E} \times \vec{H} * d\vec{A} \qquad (13)$$

and can be written, as follows:

$$P_s = P_{rad} + j2\omega(W_E - W_M) \qquad (14)$$

where W_E and W_M are the energy stored in electric field and magnetic field:

$$W_E = \frac{1}{4}\int_V \varepsilon\left|\vec{E}\right|^2 dV \qquad (15)$$

$$W_M = \frac{1}{4}\int_V \mu\left|\vec{H}\right|^2 dV \qquad (16)$$

The *Q* factor is defined, as follows [1]

$$Q = \left|\frac{Im(P_s)}{Re(P_s)}\right| = \frac{2\omega(W_E - W_M)}{P_{rad}} \tag{17}$$

Therefore, the Q-factor directly stems from the Poynting theorem and is simply obtained by the ratio of imaginary and real part of complex power (surface integral over complex Poynting vector). Thus, numerator represents active power in the notation of circuit theory, or average radiated power in the notation of the electromagnetic field theory, while the denominator pertains to the reactive power or measure of the energy stored in the electric and magnetic field, respectively.

3. Quality factor Q of Hertz Dipole in Free Space

Hertz dipole of physical length Δl, as electrically small antenna (ESA) represents the simplest radiating system. The geometry of Hertz dipole in free space with uniform current I_0 along the wire is shown in Fig 1.

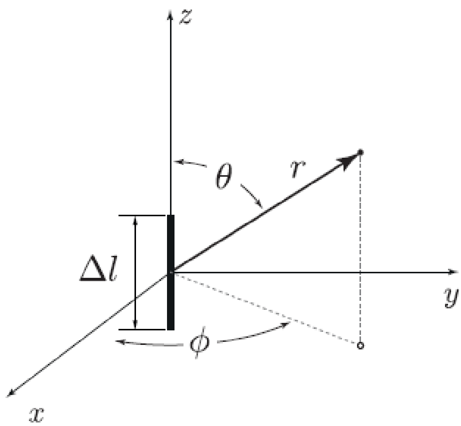

Fig. 1. Hertz dipole in free space

The complete electromagnetic field components radiated by Hertz dipole in free space are:

$$E_r(r,\theta) = Z_0 \frac{I_0 \Delta l}{2\pi r^2}\left[1 + \frac{1}{jkr}\right] e^{-jkr} \cos\theta \tag{18}$$

$$\boldsymbol{E_\theta = jZ_0 \frac{kI_0 \Delta l}{4\pi r}\left(1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right) e^{-jkr} \sin\theta} \tag{19}$$

$$\boldsymbol{H_\phi = j\frac{kI_0\Delta l}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\sin\theta} \qquad (20)$$

where k is a phase constant, and Z_0 is the free space impedance.
Note that ESA implies an antenna inside a sphere of radius a =1/k. the minimum radius of a sphere, a, which encloses a lossless antenna, is related to the maximum quality factor of the antenna, Q. As it is well-known, ESA positioned within a given volume has relatively small value of Q which corresponds to a limit of its impedance bandwidth.
The complex Poynting vector (6) for Hertz dipole is:

$$\vec{S} = \frac{1}{2}\left(\vec{E}\times\vec{H}*\right) = \frac{I^2(\Delta l)^2}{32\pi^2}\omega\mu\frac{(kr)^3-j}{r^5k^2}\sin^2\theta\cdot\vec{e}_r - j\frac{I^2(\Delta l)^2}{32\pi^2}\frac{1+r^2k^2}{r^5k}\sin(2\theta)\cdot\vec{e}_\theta \qquad (21)$$

and, consequently, by integrating (21) one obtains the apparent power for r=a:

$$P_s = \int_0^{2\pi}\int_0^{\pi}\frac{I^2(\Delta l)^2}{32\pi^2}\omega\mu\frac{(kr)^3-j}{r^5k^2}\sin^2\theta\cdot r^2\sin\theta\, d\theta d\phi = \frac{I^2(\Delta l)^2}{12\pi}\omega\mu\left(k-j\frac{1}{k^2a^3}\right) \qquad (22)$$

The Q factor as defined by (17) is then [1]:

$$Q = \left|\frac{Im(P_s)}{Re(P_s)}\right| = \frac{2\omega(W_E-W_M)}{P_{rad}} = \frac{\frac{I^2(\Delta l)^2}{12\pi k^2a^3}\omega\mu}{\frac{I^2(\Delta l)^2}{12\pi}k\omega\mu} = \frac{1}{k^3a^3} \qquad (23)$$

Note that the minimum value radius of a sphere a is related to the maximum value of quality factor of the antenna Q.

Conclusion

The paper reviews some fundamental concepts of the energy stored in the antenna electromagnetic field pertaining to the fundamental limits corresponding to the antenna size. Of particular interest is to analyze a correlation between electric/physical dimensions of the antenna and Q-factor. Maxwell equations can be obtained from the least action principle in classical electromagnetics and the general law of the energy conservation in the macroscopic electromagnetic field can be readily derived from two curl Maxwell equations. Finally, the quality factor (Q-factor) is obtained from the Poyinting theorem as the ratio of imaginary and real part of complex power (surface integral over complex Poynting vector) for the case of Hertz dipole in free space.

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O POHRANJENOJ ENERGIJI U ELEKTROMAGNETSKOM POLJU ANTENE I PRIDRUŽENIM OGRANIČENJIMA U PRIJENOSU INFORMACIJE

Sažetak: U radu se razmatraju fundamentalni koncepti očuvanja energije u elektromagnetskom polju u vidu Hamiltonovog principa u elektromagnetizmu i Poyntingovog teorema, s posebnim fokusom na analizu poveznice energije pohranjene u elektromagnetskom polju antene te fundamentalnih ograničenja koja proizlaze iz dimenzija antene. Posebno je od interesa prodiskutirati korelacije između električkih i fizičkih dimenzija antene te njenog faktora dobrote i efikasnosti zračenja što je ključno za procjenu maksimalnog mogućeg prijenosa informacije. Provedena je demonstracija izloženih teorijskih koncepata na primjeru Hertzovog dipola.

Ključne riječi: Hamiltonov princip, Poyntingov teorem, pohranjena energija, temeljna ograničenja antena

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